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Stiffness Exponent, Number of Pure States, and Almeida–Thouless Line in Spin-Glasses

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We show how an inequality for the stiffness exponent for spin-glass models proposed by Fisher and Huse could be violated. We analyze their derivation and point out that their scaling arguments do not apply to investigations of the difference between systems with periodic and antiperiodic boundary conditions. As a consequence, the possibility remains open that in sufficiently high dimensions an infinite number of pure states exists and that an Almeida–Thouless line —spin-glass transition in a field—occurs, as is predicted in competing theories.

KEY WORDS: Spin-glasses; number of pure states; Almeida–Thouless line.

Fisher and Huse[1–4] have proposed a theory of short-range spin-glass models which strongly differs from earlier attempts, in particular in high dimensions. In contrast to earlier theories, based on percolation of frustration[5] or on an expansion around the infinite-range Sherrington–Kirkpatrick (SK) model, which predict infinitely many pure states in sufficiently high dimension, Fisher and Huse claim that in *any* dimension, from three upward, the number of pure (extremal Gibbs) states[2,7] is two, and that there is no Almeida–Thouless line[8] (transition in a field). Moreover, they cast serious doubt on the applicability of SK-like concepts by showing that in finite dimensions the famous Parisi function[9] $p(q)$ is strongly dependent on boundary conditions, does not describe the number of pure states, and hence cannot be expected to describe the physics of the system.

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The main ingredient in their argument against many states is an inequality for the stiffness exponent θ. If a droplet or domain wall of linear size L exists in the system, its energy will typically be $O(L^\theta)$. The Fisher–Huse prediction is

$$\theta < \frac{d-1}{2} \tag{1}$$

From this inequality they argue that in every dimension there will be at most two pure Gibbs states and that an external magnetic field will destroy the transition—there will be no Almeida–Thouless line. In this paper we argue against the general validity of inequality (1).

In deriving this inequality the following arguments are used.[2,3] Consider a random-bond Ising Hamiltonian with symmetrically distributed coupling constants. Let us take two realizations of the bonds which are identical except in one L^{d-1} sheet (finite flat piece) within a $(d-1)$-dimensional hyperplane. All bonds in the sheet which are pointing in the direction perpendicular to the sheet have opposite signs in the two realizations. We consider the difference in ground-state energy or free energy between the two systems. If the differences in the ground-state energies due to flipping two bonds were independent or weakly dependent at large distances, by the central limit theorem the total energy difference would be

$$\Delta E = O(L^{(d-1)/2}) \tag{2}$$

If (1) holds, we have the opposite case, and there would be some long-range correlation between the effects of bonds that are far apart. The scaling arguments of ref. 3 predict[3] that the energy difference between the two ground states in that case would satisfy

$$\Delta E = O(L^{d-1-\theta}) \tag{3}$$

These scaling arguments can be expected to apply to a large sheet inside an infinite system. In this case, the difference between the two ground-state *configurations* could be of essentially two sorts. If (1) holds, then the formation of a large, approximately isotropic "droplet" is energetically favored and there would be one large region of spins on one side of the sheet, in which the spins in the two ground-state configurations point in opposite directions (the droplet). The energy of such a droplet of linear size L_1 is $O(L_1^\theta)$, and, on the other hand, because of Eq. (3), it is equal to $O(L^{d-1-\theta})$. Therefore,

$$L_1 = O(L^{(d-1-\theta)/\theta}) \tag{4}$$

[3] See especially Eqs. (A6), (A23), and following. We need not require that $B \ll L$.

so $L_1 \gg L$. The sheet is part of the boundary of the droplet and L_1 is the distance over which the influence of changing the bonds in the sheet propagates.

In the other case, when (1) is violated, the change in the ground-state configuration would consist of a large number $O[(L/l)^{d-1}]$ of small (half) droplets of typical linear size l on both sides of the sheet. Thus, the two ground-state configurations differ in a highly *an*isotropic region, namely the $L^{d-1} \times l$ region along the sheet. As the signs of the energy changes would be independent between different droplets, the total energy change would be

$$\Delta E = O\left[\left(\frac{L}{l}\right)^{(d-1)/2} l^{\theta}\right] = O(L^{(d-1)/2}) \tag{5}$$

A large droplet would give rise to an energy change $\Delta E = O(L^{\theta})$, which is more than the $O(L^{(d-1)/2})$ predicted by (5), and hence will not occur. One can think of l as a kind of correlation length, describing how far the influence of changing the bonds in the sheet propagates in the direction perpendicular to the sheet. Every droplet now has part of its boundary on (part of) the sheet. Together they cover the whole sheet.

Note that, whether (1) holds or not, there is no contribution to ΔE from bonds in the sheet itself.

The application which Fisher and Huse give, however, considers a sheet within a cylindrical system (periodic or antiperiodic boundary conditions in the direction perpendicular to the sheet, free boundary conditions in the other $d-1$ directions). The system is supposed to be of linear size L in the directions of the sheet, and also in the direction perpendicular to the sheet. It is shown that a violation of (1), together with the scaling argument, leads to the result that the influence of the flipping of the bonds in the sheet cannot extend over a distance L in the periodic direction. Since, when switching from periodic to antiperiodic boundary conditions, a domain wall must occur in one of the ground-state configurations at an essentially random position, this leads to a contradiction.

The flaw of this argument consists in the fact that the scaling arguments on which it is based do not apply to the cylindrical arrangement. Indeed, this can be seen in different ways. One way is to note that the used scaling assumptions amount to a (correlated) randomness assumption for the boundary term. The randomness may be independence (as in ref. 1) or weak dependence of the boundary conditions with respect to the interactions giving rise to the behavior as described in Eq. (3). (Problems connected with the use of random boundary conditions are discussed in refs. 10–12.) The point which is overlooked when this argument is given in refs. 1 and 3 is that, although one obtains a correct upper bound

for the ΔE, if $\theta > (d-1)/2$ there may be no large excitation at all in the system. For example, in a two-dimensional Ising ferromagnet, where $\theta = 1$ and (1) does not hold, no domain walls occur if one imposes random–antirandom boundary conditions on an $L \times L$ square; rather, the ground-state configuration changes between the "+" and the "−" configurations with probability 1/2. If N_b is the number of bonds not connected to the boundary, both "+" and "−" have energy $-N_b + O(\sqrt{L})$, which is smaller than the $-N_b + 2L + O(\sqrt{L})$ of an interface state. One of the two will have minimal energy; which one depends on the—random—boundary condition. If the sum of all the terms on the right-hand side of the boundary exceeds in absolute value the sum of all terms on the left, flipping all spins on the right boundary will cause the whole system to flip to obtain the new ground state. In the opposite case, which is of course equally probable, the ground-state configuration remains the same. On the other hand, in the three-dimensional Ising spin-glass, $\theta \approx 0.2$, Eq. (1) holds, and random–antirandom boundary conditions give rise to a domain wall.[6] This numerically most often studied case is the prime example where the Fisher–Huse description presumably is indeed valid.

However, in the periodic–antiperiodic case there *always* is a large-scale excitation, namely the domain wall. Therefore, it obeys the scaling argument *only* in the case when this argument allows for a domain wall, which is when inequality (1) holds. Therefore, the arguments of refs. 1 and 3 in favor of $\theta < (d-1)/2$ are circular: they *assume* the validity of scaling arguments for the domain wall forcing periodic–antiperiodic boundary conditions, but this validity only holds under the condition that $\theta < (d-1)/2$, which is the condition they claim to show.

Another way of visualizing what can go wrong is to consider a cylindrical system which has different lengths in the sheet directions and in the perpendicular direction. As only the *minimal*-energy domain wall will occur, the larger the system is in the perpendicular direction, the more choices there are to put in a domain wall, and thus the smaller energy this minimal-energy domain wall will have. According to ref. 3, Eq. (2.5), the density of low-energy domain walls will not be zero and if the system is too large in the perpendicular direction, the domain wall will have an energy which is lower than the "typical" $O(L^\theta)$. The point which is overlooked in the appendix of ref. 3, is that an $L^{d-1} \times L$ system may already be too big in the perpendicular direction to get a "typical" domain wall. As remarked before, the influence of flipping bonds in a sheet in an infinite system propagates over a distance which is only dependent on θ and the size of the sheet. In the cylindrical arrangement this distance is just the length of the system in the perpendicular direction, which is imposed from the outside, but does not depend on θ nor on the size of the sheet.

Summarizing, if $\theta > (d-1)/2$, the only conclusions one can draw are that random boundary conditions cannot generate a domain wall, and domain wall forcing boundary conditions cannot be considered random and cannot satisfy scaling. The contradiction found by Fisher and Huse only reflects the inappropriate use of scaling arguments rather than the impossibility of violating (1). By default, this leaves open the possibility of infinitely many pure phases and an Almeida–Thouless line in high dimensions, as predicted in refs. 5 and 6.

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